

Derived Kan extension for strict polynomial functors

Marcin Chałupnik *

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Abstract

We construct and investigate basic properties of adjoint functors to the precomposition functor in the category of strict polynomial functors. As applications we get the proof of Touzé's conjecture on Ext groups between twisted functors and we obtain various known Ext-computations in a much simpler way.

1 Introduction

In the present article we construct the (right and left) adjoint functors $\mathbf{K}_A^r, \mathbf{K}_A^l$ to the precomposition functor in the derived category of the category of strict polynomial functors and establish their basic properties. Our main result is a computation of the unit of the (right) adjunction for the precomposition with the iterated Frobenius twist functor. This leads to the proof of a stronger version of Touzé's Collapsing Conjecture (see [To, Sect. 8], this article highly inspired the present note). Namely, we obtain

Corollary 3.3 *For any strict polynomial functors F, G and $i \geq 0$, we have a natural in F, G isomorphism of graded spaces*

$$\mathrm{Ext}_{\mathcal{P}}^*(F^{(i)}, G^{(i)}) \simeq \mathrm{Ext}_{\mathcal{P}}^*(F, G \circ (- \otimes A_i))$$

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where A_i is a graded space which one-dimensional in even degrees smaller than $2p^i$ (p is a characteristic of a ground field) and trivial elsewhere.

We also compute $\mathbf{K}_{I(i)}^r$ for a certain class Schur functors (Prop. 3.7), which allows us to re-obtain the Ext-computations from [C1, C3] in a much simpler way (Cor. 3.4, Cor. 3.5, Cor. 3.8).

2 Precomposition and its adjoints

Let \mathcal{V} (resp. \mathcal{V}^{gr}) be the category finite-dimensional vector spaces (resp. graded, finite-dimensional in each degree, vector spaces) over a fixed field \mathbf{k} . We work in the category \mathcal{P}_d of homogeneous strict polynomial functors of degree d over \mathbf{k} and its bounded derived category \mathcal{DP}_d (i.e. \mathcal{DP}_d is the category of finite complexes of objects of \mathcal{P}_d modulo quasi-isomorphisms). For $A \in \mathcal{P}_s$ let $\mathbf{C}_A : \mathcal{P}_d \rightarrow \mathcal{P}_{ds}$ be the functor of taking precomposition with A i.e. $\mathbf{C}_A(F)(V) := F(A(V))$. Since \mathbf{C}_A is an exact functor between abelian categories, it extends degree-wise to their derived categories and we shall, slightly abusing notation, denote both functors by \mathbf{C}_A . Our task in this section is to construct functors

$$\mathbf{K}_A^r, \mathbf{K}_A^l : \mathcal{DP}_{ds} \rightarrow \mathcal{DP}_d$$

which are respectively the right and left adjoint to \mathbf{C}_A . We call these functors respectively the right and left (derived) Kan extensions by analogy with topology. In fact, the existence of adjoints to the precomposition is quite general categorical phenomenon, and we will see that our construction is rather tautological.

We start with recalling some standard constructions in \mathcal{P}_d . For $F \in \mathcal{P}_d$ and $U \in \mathcal{V}$ let $F_U(V) := F(U^* \otimes V)$. It is in fact a strict polynomial bifunctor of bidegree (d, d) in the sense of [FF], but unless otherwise stated, we shall regard F_U just as a functor in V treating U as a parameter. Let $I^d, D^d, S^d, \Lambda^d \in \mathcal{P}_d$ denote respectively the tensor, divided, symmetric and exterior power functors. By a suitable version of the Yoneda lemma we get for any $F \in \mathcal{P}_d$ a natural in F and U isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d}(D_U^d, F) \simeq F(U),$$

hence D_U^d form a family of projective generators of \mathcal{P}_d . Analogously, we have

$$\mathrm{Hom}_{\mathcal{P}_d}(F, S_U^d) \simeq (F(U))^* \simeq F^\#(U^*),$$

where $F^\#$ stands for the Kuhn dual of F (i.e. $F^\#(V) := (F(V^*))^*$). Thus S_U^d form a family of injective cogenerators of \mathcal{P}_d .

Now we are ready to define our adjoints. For $F \in \mathcal{DP}_{ds}$ we put

$$\mathbf{K}_A^r(F)(V) := \mathrm{Hom}_{\mathcal{DP}_{ds}}(\mathbf{C}_A(D_V^d), F),$$

and dually

$$\mathbf{K}_A^l(F) := (\mathbf{K}_A^r(F^\#))^\#.$$

Since, as we have mentioned, D_V^d is also a strict polynomial functor of degree d in V , $\mathbf{K}_A^r(F), \mathbf{K}_A^l \in \mathcal{DP}_d$. Thus we get the functors $\mathbf{K}_A^r, \mathbf{K}_A^l : \mathcal{DP}_{ds} \rightarrow \mathcal{DP}_d$.

We start with computing the values of $\mathbf{K}_A^r, \mathbf{K}_A^l$ on cogenerators.

Proposition 2.1 $\mathbf{K}_A^r(S_U^{ds}) = S_{A(U)}^d$ and dually $\mathbf{K}_A^l(D_U^{ds}) = D_{A(U)}^d$.

Proof: Since $(D^d(W))^* \simeq S^d(W^*)$ for any space W , we get

$$\begin{aligned} \mathbf{K}_A^r(S_U^{ds})(V) &= \mathrm{Hom}_{\mathcal{DP}_{ds}}(\mathbf{C}_A(D_V^d), S_U^{ds}) \simeq (\mathbf{C}_A(D_V^d)(U))^* = (D^d(V^* \otimes A(U)))^* \simeq \\ &S^d((A(U))^* \otimes V) = S_{A(U)}^d(V). \end{aligned}$$

The formula for $\mathbf{K}_A^l(D_U^{ds})$ is proved analogously. ■

From this computation we deduce our adjunction in a purely formal manner.

Theorem 2.2 $\mathbf{K}_A^r, \mathbf{K}_A^l : \mathcal{DP}_{ds} \rightarrow \mathcal{DP}_d$ are respectively the right and left adjoint functor to $\mathbf{C}_A : \mathcal{DP}_d \rightarrow \mathcal{DP}_{ds}$.

Proof: We only prove the right adjunction, the proof for the left one is analogous. Since S_U^{ds} cogenerate \mathcal{P}_{ds} it suffices to establish a natural in $F \in \mathcal{DP}_d$ and $U \in \mathcal{V}$ isomorphism

$$\mathrm{Hom}_{\mathcal{DP}_{ds}}(\mathbf{C}_A(F), S_U^{ds}) \simeq \mathrm{Hom}_{\mathcal{DP}_d}(F, \mathbf{K}_A^r(S_U^{ds})).$$

At the left-hand side we have

$$\mathrm{Hom}_{\mathcal{DP}_{ds}}(\mathbf{C}_A(F), S_U^{ds}) \simeq (F(A(U)))^*,$$

at the right-hand side, by Prop. 2.1, we get

$$\mathrm{Hom}_{\mathcal{DP}_d}(F, \mathbf{K}_A^r(S_U^{ds})) \simeq \mathrm{Hom}_{\mathcal{DP}_d}(F, S_{A(U)}^d) \simeq (F(A(U)))^*,$$

which proves our statement. ■

In some important cases the functors $\mathbf{K}_A^r, \mathbf{K}_A^l$ may be described more explicitly. For a Young diagram λ of weight d we put $D^\lambda := D^{\lambda_1} \otimes \dots \otimes D^{\lambda_k}$ and we define $S^\lambda, \Lambda^\lambda$ analogously.

Proposition 2.3 *Assume that for any Young subgroup $\Sigma_\lambda \subseteq \Sigma_d$, the natural embedding $D^\lambda \longrightarrow I^d$ induces an isomorphism on Ext-groups*

$$\mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(D^\lambda), F) \simeq (\mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(I^d), F))_{\Sigma_\lambda}.$$

Then

$$\mathbf{K}_A^r(F)(V) \simeq V^{\otimes d} \otimes_{\Sigma_d} \mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(I^d), F),$$

in particular $\mathbf{K}_A^r(F)$ is formal.

Dually, if $\mathrm{Ext}_{\mathcal{P}_{ds}}^*(F, \mathbf{C}_A(S^\lambda)) \simeq (\mathrm{Ext}_{\mathcal{P}_{ds}}^*(F, \mathbf{C}_A(I^d)))_{\Sigma_\lambda}$, then

$$\mathbf{K}_A^l(F)(V) = \mathrm{Hom}_{\Sigma_d}(\mathrm{Ext}_{\mathcal{P}_{ds}}^*(F, \mathbf{C}_A(I^d)), V^{\otimes d}),$$

in particular $\mathbf{K}_A^l(F)$ is formal.

Proof: Again we focus on the formula for \mathbf{K}_A^r leaving the case of \mathbf{K}_A^l to the reader. We consider the composition

$$\alpha : \mathrm{Hom}_{\mathcal{P}_d}(D_V^d, I^d) \otimes \mathrm{Hom}_{\mathcal{DP}_{ds}}(\mathbf{C}_A(I^d), F) \longrightarrow$$

$$\mathrm{Hom}_{\mathcal{P}_{ds}}(\mathbf{C}_A(D_V^d), \mathbf{C}_A(I^d)) \otimes \mathrm{Hom}_{\mathcal{DP}_{ds}}(\mathbf{C}_A(I^d), F) \longrightarrow \mathrm{Hom}_{\mathcal{DP}_{ds}}(\mathbf{C}(D_V^d), F)$$

where the first arrow is the precomposition with A tensored with the identity and the second is the Yoneda composition. We shall look at the effect of this transformation on the Ext-groups. We take a partition λ of $n = \dim(V)$ and we choose some basis of V . Since by the Yoneda Lemma $\mathrm{Hom}_{\mathcal{P}_d}(D_V^d, I^d) \simeq V^{\otimes d}$, the subspace of $\mathrm{Hom}_{\mathcal{P}_d}(D_V^d, I^d) \otimes \mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(I^d), F)$ on which the standard torus in V acts with a weight λ may be identified with $(\mathbf{k} \otimes_{\Sigma_\lambda} \mathbf{k}[\Sigma_d]) \otimes \mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(I^d), F)$. On the other hand the λ -weight subspace of $\mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(D_V^d), F)$ may be identified with $\mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(D^\lambda), F)$. Under these identifications α_* is just induced by the embedding $D^\lambda \longrightarrow I^d$. Thus, by our assumption α induces an epimorphism on Ext-groups, which

shows, since the source is formal, that the target is formal too. Finally, since α_* factorizes to the isomorphism

$$(\mathbf{k} \otimes_{\Sigma_\lambda} \mathbf{k}[\Sigma_d]) \otimes_{\Sigma_d} \mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(I^d), F) \simeq \mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(D^\lambda), F),$$

by gathering up these isomorphisms for all λ we get an isomorphism

$$V^{\otimes d} \otimes_{\Sigma_d} \mathrm{Ext}^*(\mathbf{C}_A(I^d), F) \simeq \mathrm{Ext}_{\mathcal{P}_{ds}}^*(\mathbf{C}_A(D^\lambda), F).$$

■

3 Application to the Frobenius twist

In this section we restrict our attention to the case of a base field \mathbf{k} of positive characteristic p and $A = I^{(i)}$ (the i -th Frobenius twist), but we begin with making a simple observation which applies to a somewhat wider class of functors.

Proposition 3.1 *Assume that for $A \in \mathcal{P}_s$ we have an isomorphism of bifunctors $A(V \otimes W) \simeq A(V) \otimes A(W)$. Then for any $U \in \mathcal{V}^{gr}$, and $F \in \mathbf{DP}_{ds}$, $K_A^r(F_U) \simeq (K_A^r(F))_{A(U)}$ and the same holds for \mathbf{K}_A^l .*

Proof: It suffices to establish this formula for \mathbf{K}_A^r and $F = S_W^{ds}$. By Proposition 2.1

$$\begin{aligned} \mathbf{K}_A^r((S_W^{ds})_U) &= \mathbf{K}_A^r(S_{W \otimes U}^{ds}) \simeq S_{A(W \otimes U)}^d \simeq S_{A(W) \otimes A(U)}^d = (S_{A(W)}^d)_{A(U)} \simeq \\ &\simeq (\mathbf{K}_A^r(S_W^d))_{A(U)}. \end{aligned}$$

■

From now on we denote $\mathbf{C}_{I^{(i)}}$ just by \mathbf{C}_i and $\mathbf{K}_{I^{(i)}}^r$ by \mathbf{K}_i (also all results of this section have obvious counterparts for $\mathbf{K}_{I^{(i)}}^l$ but we leave their formulation to the reader). We start with computing $\mathbf{K}_i(F^{(i)})$ ($F^{(i)}$ stands for $\mathbf{C}_{I^{(i)}}(F)$), which proves a stronger version of Touzé's Collapsing Conjecture. Let A_i be a graded space which is one-dimensional in all nonnegative even degrees smaller than $2p^i$ and trivial elsewhere. Then F_{A_i} is a graded strict polynomial functor which we regard as an object in \mathcal{DP}_d (with a trivial differential).

Theorem 3.2 *For any $F \in \mathcal{P}_d$, $\mathbf{K}_i(F^{(i)}) \simeq F_{A_i}$. In particular $\mathbf{K}_i(F^{(i)})$ is formal.*

Proof: Let $T^{d,i}$ mean the Troesch resolution of $D^{d(i)}$. Explicitly, $T^{d,i} = D_{S_i}^{dp^i}$ where S_i is a graded space which is one-dimensional in all nonnegative degrees smaller than p^i and trivial elsewhere (for description of the grading and differential on $T^{d,i}$ (or rather the dual injective resolution of $S^{d(i)}$) see [Tr, To]). Thus $T_V^{d,i}$ is a projective resolution of $\mathbf{C}_i(D_V^d)$. Hence $\mathbf{K}_i(F^{(i)})(V) = \text{Hom}_{\mathcal{PP}_{dp^i}}(T_V^{d,i}, F^{(i)})$. But it follows from Touzé's observation [To, Lemma 4.1] that $\text{Hom}_{\mathcal{PP}_{dp^i}}^{\text{odd}}(T_V^{d,i}, F^{(i)}) = 0$. Hence $\mathbf{K}_i(F^{(i)})$ is formal and all we need is to compute its cohomology. To this end we use again [To, Lemma 4.1] and get

$$H^*(\text{Hom}_{\mathcal{PP}_{dp^i}}(T_V^{d,i}, F^{(i)})) = \text{Hom}_{\mathcal{P}_{dp^i}}^{\text{ev}}(T_V^{d,i}, F^{(i)}) \simeq \text{Hom}_{\mathcal{P}_{dp^i}}^*(D_{S_i \otimes V}^{dp^i}, F^{(i)}) \simeq F_{A_i}(V).$$

■

This, by adjunction and taking cohomology, gives

Corollary 3.3 *For any $F, G \in \mathcal{P}_d$ we have a natural in F, G isomorphism of graded spaces*

$$\text{Ext}_{\mathcal{P}_{dp^i}}^*(F^{(i)}, G^{(i)}) \simeq \text{Ext}_{\mathcal{P}_d}^*(F, G_{A_i}).$$

This, as was pointed out in [To; Sect. 4, Sect. 7], allows one to quickly re-obtain the Ext-computations of [C1]. We recall from [C1, Section 5] that for $F \in \mathcal{P}_d$ and a Young diagram λ of weight d we define a d -functor \tilde{F}^λ as the component in $F(V_1 \oplus \dots \oplus V_d)$ of multidegree λ . Then we put $F^\lambda(V) := \tilde{F}^\lambda(V, \dots, V)$. Now it follows from the Yoneda lemma that $\text{Hom}_{\mathcal{P}_d}(D^\lambda, F) \simeq F^\lambda(\mathbf{k})$. Thus, by Cor. 3.3 we get [C1, Cor. 5.1]

Corollary 3.4 *For any $F \in \mathcal{P}_d$ and λ of weight d*

$$\text{Ext}_{\mathcal{P}_{dp^i}}^*(D^{\lambda(i)}, F^{(i)}) \simeq F^\lambda(A_i).$$

Also the rest of computations of [C1] may be obtained with the aid of Cor. 3.3.

Namely, let W_μ, S_λ be respectively Weyl and Schur functors associated to Young diagrams μ, λ of weight d , and let s_μ, s_λ be appropriate symmetrizations [C1, Section 3]. Then we have [C1, Th. 6.1]

Corollary 3.5

$$\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(W_\mu^{(i)}, S_\lambda^{(i)}) \simeq s_\mu(s_\lambda(A_i^{\otimes d} \otimes \mathbf{k}[\Sigma_d])) \simeq s_\lambda(s_\mu(A_i^{\otimes d} \otimes \mathbf{k}[\Sigma_d])).$$

Proof: Cor. 3.3 allows us to replace $\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(W_\mu^{(i)}, S_\lambda^{(i)})$ with $\mathrm{Ext}_{\mathcal{P}_d}^*(W_\mu, (S_\lambda)_{A_i})$, i.e. in the terminology of [C1] we only need to prove the “additive version” of the formula. This is rather straightforward and was accomplished in the proofs of [C1; Th. 4.4, Th. 6.1]. ■

Now we show that the functor \mathbf{K}_i can be effectively used in computations of $\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(F^{(i)}, G)$ also for nontwisted G .

Proposition 3.6 *Assume that for any Young diagram ρ of weight d , the embedding $D^{\rho(i)} \rightarrow I^{d(i)}$ induces an isomorphism*

$$\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(D^{\rho(i)}, S_\lambda) \simeq (\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda))_{\Sigma_\rho}.$$

Then

$$\mathbf{K}_i(S_\lambda) \simeq I^d \otimes_{\Sigma_d} \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda).$$

If additionally $\mathbf{K}_i(S_\lambda)$ has a “good filtration” i.e. such that its associated object is a sum of Schur functors, then for any Young diagram μ of weight d ,

$$\mathrm{Ext}_{\mathcal{P}_{dp^{i+j}}}^*(W_\mu^{(i+j)}, S_\lambda^{(j)}) \simeq s_\mu(\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda) \otimes (A_j^{(i)})^{\otimes d}).$$

Proof: The first part of the proposition is just Prop. 2.3. We turn then to the proof of the second part. By Cor. 3.3 and Prop. 3.1 we get

$$\mathrm{Ext}_{\mathcal{P}_{dp^{i+j}}}^*(W_\mu^{(i+j)}, S_\lambda^{(j)}) \simeq \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(W_\mu^{(i)}, (S_\lambda)_{A_j^*}) \simeq \mathrm{Ext}_{\mathcal{P}_d}^*(W_\mu, \mathbf{K}_i((S_\lambda)_{A_j^*})) \simeq$$

$$\mathrm{Ext}_{\mathcal{P}_d}^*(W_\mu, \mathbf{K}_i(S_\lambda)_{A_j^{(i)*}}) \simeq \mathrm{Ext}_{\mathcal{P}_d}^*(W_\mu, (I^d \otimes_{\Sigma_d} \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda))_{A_j^{(i)}}).$$

Let

$$0 \longrightarrow D^{\mu^k} \longrightarrow \dots \longrightarrow D^{\mu^1} \xrightarrow{\phi} D^\mu \longrightarrow W_\mu \longrightarrow 0$$

be a resolution of W_μ by sums of products of divided powers. To simplify notation we denote $(I^d \otimes_{\Sigma_d} \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda))_{A_j^{(i)}}$ by X . Since $\mathrm{Ext}_{\mathcal{P}_d}^{>0}(W_\mu, S_\rho) = 0$ for any ρ , we have $\mathrm{Ext}_{\mathcal{P}_d}^{>0}(W_\mu, X)$ by the Decomposition Formula and our assumption on existence of a good filtration on $\mathbf{K}_i(S_\lambda)$. Hence, the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(W_\mu, X) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(D^\mu, X) \xrightarrow{\phi^*} \dots \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(D^{\mu^k}, X) \longrightarrow 0$$

is exact. Thus

$$\mathrm{Hom}_{\mathcal{P}_d}(W_\mu, X) = \ker(\mathrm{Hom}_{\mathcal{P}_d}(D^\mu, X) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{P}_d}(D^{\mu^1}, X)).$$

We describe this kernel by the arguments used in the proof of [C1, Th. 6.1]. In order to further simplify notation we put $Y := \mathrm{Hom}_{\mathcal{P}_d}(I^d, X)$. Then by the fact that X has a good filtration and [C1, Lemma 6.2] we get

$$\mathrm{Hom}_{\mathcal{P}_d}(D^\mu, X) \simeq (Y)_{\Sigma_\mu}, \quad \mathrm{Hom}_{\mathcal{P}_d}(D^{\mu^1}, X) \simeq (Y)_{\Sigma_{\mu^1}}.$$

This, as it was shown in the proof of [C1, Th. 6.1], allows us to rewrite this kernel as

$$\ker(s^\mu(Y) \xrightarrow{\phi(Y)} s^{\mu^1}(Y)) = s_\mu(Y)$$

(we identified here ϕ with the corresponding transformation between symmetrizations). Then it remains to observe that

$$\begin{aligned} Y &= \mathrm{Hom}_{\mathcal{P}_d}(I^d, (I^d \otimes_{\Sigma_d} \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda))_{A_j^{(i)}}) \simeq \\ &\simeq \mathrm{Hom}_{\mathcal{P}_d}(I^d, \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda)) \otimes (A_j^{(i)})^{\otimes d} \simeq \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda) \otimes (A_j^{(i)})^{\otimes d}, \end{aligned}$$

we use here two general facts: that $\mathrm{Hom}_{\mathcal{P}_d}(I^d, F_U) \simeq \mathrm{Hom}_{\mathcal{P}_d}(I^d, F) \otimes (U^*)^{\otimes d}$ for any $F \in \mathcal{P}_d$ and $U \in \mathcal{V}^{gr}$, and that $\mathrm{Hom}_{\mathcal{P}_d}(I^d, I^d \otimes_{\Sigma_d} M) \simeq M$ for any Σ_d -module M ; and at last we identify $A_j^{(i)}$ with its linear dual. This proves the second part of the Proposition. ■

Thus the problem of computing $\mathbf{K}_i(S_\lambda)$ is essentially reduced to that of describing $\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_\lambda)$ as a graded Σ_d -module. This goal was achieved in one important special case. Let $F_k(\lambda)$ be a Young diagram with a trivial p -core and the p -quotient with only nontrivial k -th diagram which is λ , and let $F_k^{i+1}(\lambda) := F_k(F_k^i(\lambda))$ (see [C2, Section 5]).

Proposition 3.7 *For any Young diagram λ of weight d , $i > 0$, $0 \leq k \leq p-1$, we have an isomorphism of graded Σ_d -modules*

$$\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_{F_k^i(\lambda)}) \simeq Sp_\lambda[h_k^i],$$

where Sp_λ is the Specht module associated to λ [JK, Chap. 7.1] and $[h_k^i]$ is a suitable shift of grading [C3, Th. 4.4].

Moreover, $S_{F_k^i(\lambda)}$ satisfies both assumptions of Prop. 3.6, hence

$$\mathbf{K}_i(S_{F_k^i(\lambda)}) \simeq S_\lambda[h_k^i].$$

Remark: These facts follow from [C3, Th. 4.4] but we shall prove them independently, since, as we will see, [C3, Th. 4.4] may be deduced from our Prop. 3.7. This way we will obtain the Ext-computations of [C3] in a much simpler way.

Proof: The proof is a rather eclectic mixture of arguments from [C3] but it is still much simpler than that of [C3, Th. 4.4] (in particular we do not refer to the Schur–de Rham complex).

We need from combinatorial machinery developed in [C3] some properties of “homological structural arrows”: $\phi : \Lambda^{F_k^i(\lambda)} \longrightarrow S_{F_k^i(\lambda)}$, $\psi : S_{F_k^i(\lambda)} \longrightarrow S^{\tilde{F}_k^i(\lambda)}$ (strictly speaking ϕ and ψ exist in a suitable localized category to which one can transport computations of the Ext-groups). It was shown in [C3, Section 3.2] that the image of the map induced by $\psi \circ$ on $\text{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, -)$ equals Sp_λ . Thus to finish the proof of the first formula it suffices to observe that $\dim(\text{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_{F_k^i(\lambda)})) = \dim(Sp_\lambda)$ [C3, p. 46].

We show that for any Young diagram ρ of weight d the embedding $D^{\rho(i)} \longrightarrow I^{d(i)}$ induces on $\text{Ext}_{\mathcal{P}_{dp^i}}^*(-, S_{F_k^i(\lambda)})$ taking the coinvariants by induction on d . By the Decomposition Formula and [C3, Fact 3.4], it suffices to show this for $\rho = (d)$. Since

$$\begin{aligned} (\text{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_{F_k^i(\lambda)}))_{\Sigma_d} &\simeq (Sp_\lambda)_{\Sigma_d}[h_k^i] \simeq (\text{Hom}_{\mathcal{P}_d}(I^d, S_\lambda))_{\Sigma_d}[h_k^i] \simeq \\ &\simeq \text{Hom}_{\mathcal{P}_d}(D^d, S_\lambda)[h_k^i] \simeq S_\lambda(\mathbf{k})[h_k^i], \end{aligned}$$

by [C1, Th. 6.1], we see that $\text{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, S_{F_k^i(\lambda)})_{\Sigma_d} \neq 0$ iff $\lambda = (1^d)$. Moreover for $\lambda = (1^d)$ our formula follows from [FFSS, Th. 4.5] (strictly speaking this is the case for $k = 0$, for $k > 0$ the Ext-groups are just shifted, which may be shown by applying the Littlewood–Richardson Rule to $S_{(a, 1^{dp^i-a-1})} \otimes I$). Thus it remains to show that $\text{Ext}_{\mathcal{P}_{dp^i}}^*(D^{d(i)}, S_{F_k^i(\lambda)}) = 0$ for $\lambda \neq (1^d)$. Let us first consider the case when λ contains the diagram $(2, 2)$. We will show that $\text{Ext}_{\mathcal{P}_{dp^i}}^*(D^{dp^{i-s}(s)}, S_{F_k^i(\lambda)}) = 0$ (and also that $\text{Ext}_{\mathcal{P}_{dp^i}}^*(\Lambda^{dp^{i-s}(s)}, S_{F_k^i(\lambda)}) = 0$ by induction on s . Assume these vanishings for $s - 1$ and consider the hyper-Ext spectral sequences converging to $\text{Hom}_{\mathcal{DP}_{dp^i}}(\mathbf{dR}^{dp^{i-s+1}(s-1)}, S_{F_k^i(\lambda)})$, where $\mathbf{dR}^{dp^{i-s+1}(s-1)}$ is the $(s - 1)$ -twisted dual de Rham complex:

$$0 \longrightarrow \Lambda^{dp^{i-s+1}(s-1)} \longrightarrow \dots \longrightarrow D^{dp^{i-s+1}(s-1)} \longrightarrow 0.$$

By the induction assumption and the Decomposition Formula the first spectral sequence is trivial. Hence the second spectral sequence converges to 0. But by the induction assumption and the Decomposition Formula all but the outer ones terms in its E_2 -page are trivial. Thus we get the shift in grading

$$\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(D^{d(i)}, S_{F_k^i(\lambda)}) \simeq \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(\Lambda^{d(i)}, S_{F_k^i(\lambda)})[dp^{i-s} - 1].$$

Now we analyze in a similar manner the spectral sequence converging to $\mathrm{Hom}_{\mathcal{DP}_{dp^i}}(\mathbf{Ko}^{dp^{i-s}(s)}, S_{F_k^i(\lambda)})$, where $\mathbf{Ko}^{dp^{i-s}(s)}$ is the s -twisted dual Koszul complex:

$$0 \longrightarrow D^{dp^{i-s+1}(s-1)} \longrightarrow \dots \longrightarrow \Lambda^{dp^{i-s+1}(s-1)} \longrightarrow 0.$$

From this we get the opposite shift in grading:

$$\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(\Lambda^{d(i)}, S_{F_k^i(\lambda)}) \simeq \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(D^{d(i)}, S_{F_k^i(\lambda)})[dp^{i-s} - 1],$$

which shows that the considered groups are trivial. It remains to consider the case of λ not containing $(2, 2)$ i.e. of the form $(a, 1^{d-a})$. We first take $\lambda = (2, 1^{d-2})$ and apply $\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(D^{d(i)}, -)$ to the Littlewood–Richardson filtration on $S_{F_k^i((1^{d-1}))} \otimes S_{F_k^i((1))}$. After neglecting terms with trivial Ext-groups, we get the long exact sequence

$$\rightarrow \mathrm{Ext}^*(D^{d(i)}, S_{F_k^i((2, 1^{d-2}))}) \rightarrow \mathrm{Ext}^*(D^{d(i)}, S_{F_k^i((1^{d-1}))} \otimes S_{F_k^i((1))}) \rightarrow \mathrm{Ext}^*(D^{d(i)}, S_{F_k^i((1^d))}) \rightarrow .$$

By [FFSS, Th. 4.5] the right arrow is an isomorphism, thus $\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(D^{d(i)}, S_{F_k^i((2, 1^{d-2}))}) = 0$. Then by using the Littlewood–Richardson Rule for $S_{F_k^i((1^{d-a+1}))} \otimes S_{F_k^i((a-1))}$ we show inductively on a that $\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(D^{d(i)}, S_{F_k^i((a, 1^{d-a}))}) = 0$ for all $a \geq 2$.

Thus, we have shown that $S_{F_k^i(\lambda)}$ satisfies the first assumption in Prop. 3.6. Hence by Prop. 3.6 we get

$$\mathbf{K}_i(S_{F_k^i(\lambda)}) \simeq I^d \otimes_{\Sigma_d} Sp_\lambda[h_k^i] \simeq S_\lambda[h_k^i].$$

This also shows that $S_{F_k^i(\lambda)}$ satisfies the second assumption in Prop. 3.6. ■

Now we can obtain the promised Ext-computations from [C3].

Corollary 3.8 *For any μ, λ of weight d , $i > 0$, $0 \leq k \leq p-1$,*

$$\mathrm{Ext}_{\mathcal{P}_{dp^{i+j}}}^*(W_\mu^{(i+j)}, S_{F_k^i(\lambda)}^{(j)}) \simeq s_\mu(s_\lambda(A_j^{(i) \otimes d} \otimes \mathbf{k}[\Sigma_d]))[h_k^i] \simeq s_\lambda(s_\mu(A_j^{(i) \otimes d} \otimes \mathbf{k}[\Sigma_d]))[h_k^i].$$

Proof: We observe that, by Prop. 3.7, $\mathbf{K}_i(S_{F_k^i(\lambda)})$ has a good filtration. Thus by applying Prop. 3.7, Prop. 3.6 we get

$$\begin{aligned} \mathrm{Ext}_{\mathcal{P}_{dp^{i+j}}}^*(W_\mu^{(i+j)}, S_{F_k^i(\lambda)}^{(j)}) &\simeq s_\mu(Sp_\lambda \otimes A_j^{(i) \otimes d}[h_k^i]) \simeq s_\mu(s_\lambda(\mathbf{k}[\Sigma_d]) \otimes A_j^{(i) \otimes d})[h_k^i] = \\ &= s_\mu(s_\lambda(\mathbf{k}[\Sigma_d] \otimes A_j^{(i) \otimes d}))[h_k^i]. \end{aligned}$$

The fact that s_μ and s_λ commute in our situation is purely formal and follows e.g. from [C1, Th. 6.1]. ■

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